PHYS 798C Spring 2022
Lecture 23 Summary

Prof. Steven Anlage

I. THE RSJ MODEL OF A JOSEPHSON JUNCTION

In general both quasiparticles and Cooper pairs can tunnel through the barrier in a Josephson junction when a potential difference (or equivalently $d\gamma/dt \neq 0$) is present. To include this possibility we treat the circuit model of a JJ as a parallel combination of an ideal Josephson junction (that obeys the two Josephson equations) and a resistor (that obeys a generalization of Ohm's law for nonlinear resistors). The resistance will in general depend on bias voltage and temperature, $R_N = R(V,T)$. This is known as the resistively shunted junction model (RSJ).
A bias current on the JJ will in general split between the two branches and produce a total current of,
$I = I_c \sin\gamma + \frac{\Phi_0}{2\pi R_N}\frac{d\gamma}{dt}$.

More generally, if a finite frequency bias current is applied, the junction typically also has parasitic capacitance, so we add a capacitor in parallel with the ideal junction and resistor, creating the RCSJ model. The total current through the JJ can now split three ways in general,
$I = I_c \sin\gamma + \frac{\Phi_0}{2\pi R_N}\frac{d\gamma}{dt} + C\frac{d\Delta V}{dt}$, where ΔV is the voltage drop on the 3 parallel circuit elements. Using the ac Josephson equation, this can be written as,
$\frac{\hbar C}{2e}\frac{d^2\gamma}{dt^2} = [I - I_c\sin\gamma] - \frac{\hbar}{2eR_N}\frac{d\gamma}{dt}$.
This equation has the appearance of a "mass times acceleration" on the left hand side, a conservative force in square brackets, and a dissipative force (function of velocity) on the right. Let's derive a potential energy associated with the conservative force and look at the equation of motion from a power perspective. Multiply the current (I) equation by voltage ($\frac{\hbar}{2e}\frac{d\gamma}{dt}$) to get the instantaneous power equation as,
$\frac{d}{dt}\left\{\frac{1}{2}(\frac{\hbar}{2e})^2 C(\frac{d\gamma}{dt})^2 + \left[\frac{-\hbar I}{2e}\gamma - \frac{\hbar I_c}{2e}\cos\gamma\right]\right\} = -(\frac{\hbar}{2e})^2\frac{1}{R_N}(\frac{d\gamma}{dt})^2$.
The left side appears to be the time rate of change of kinetic energy plus potential energy, while the right hand side is the power dissipated in the resistor.

The potential energy associated with a Josephson junction biased by current I is therefore:
$U(\gamma) = \frac{-\hbar I}{2e}\gamma - \frac{\hbar I_c}{2e}\cos\gamma$. This is known as the tilted washboard potential. The washboard $\cos\gamma$ piece is tilted by the bias current I. The solution to the original equation is now reduced to finding the coordinate γ of a fictitious massive particle living in this potential and being subjected to driving and drag forces. The particle mass (or junction 'inertia') is proportional to the capacitance C of the junction.

From an alternative perspective, the current-biased JJ acts in a manner similar to a pendulum of mass m and length l hanging in a gravitational field $\vec{g}$. The bias current acts as an external torque on the pendulum, and a dissipative force retards the motion of the pendulum. The equation of motion for the analog pendulum is,
$\tau_a = mgl\sin\gamma + D\frac{d\gamma}{dt} + M\frac{d^2\gamma}{dt^2}$, where the angle γ is analogous to the GIPD on the JJ, the angular velocity is analogous to the voltage on the JJ, the moment of inertia M is analogous to the capacitance term in the JJ, the damping D is analogous to $1/R_N$ in the JJ, and the applied torque τ_a is analogous to the driving current on the JJ. A classic paper discussing this analogy is posted on the class web site.

II. TILTED WASHBOARD POTENTIAL

The motion of a point particle in the tilted washboard potential is a useful way to visualize the behavior of a current-biased JJ. Imagine a ball subjected to gravity moving over a corrugated one-dimensional surface with various degrees of tilt and wiggling, subjected to a drag force proportional to speed. One can think of Johnson noise in the resistor as being analogous to Brownian motion of the particle in a viscous fluid that provides the drag. They are related through the Fluctuation-Dissipation theorem.

Starting from a horizontal washboard ($I = 0$) imagine tilting it to one side as the dc current is applied. As this happens, the phase point will seek out a new minimum in the potential, and move to a coordinate

given by $\gamma = \sin^{-1}(\frac{I_{dc}}{I_c})$. At some point as the tilt is increased the phase point is unstable and will begin to run ($\frac{d\gamma}{dt} > 0$), putting the junction in to the finite-voltage state. The critical tilt comes when $\frac{dU}{d\gamma} = 0$, or when $I_{DC} = I_c$.
As the phase point moves in the case of $I > I_c$, it will speed up and slow down periodically (but not sinusoidally) in time. For strong enough driving current ($I_{dc} >> I_c$) the junction will behave like a resistor, $I_{dc} \approx \frac{\hbar}{2eR_N}\frac{d\gamma}{dt}$, or in other words, $I_{dc}R_N = \frac{\hbar}{2e}\frac{d\gamma}{dt} = \Delta V$.

III. CURRENT-VOLTAGE CHARACTERISTIC OF A JOSEPHSON JUNCTION

Since the potential is periodic, we expect the motion of the phase point to be periodic too. We can write the time-average of the voltage as,
$\left\langle \frac{d\gamma}{dt} \right\rangle \equiv \frac{2\pi}{T}$, where T is the period of the motion.
One can also write $\left\langle \frac{d\gamma}{dt} \right\rangle \equiv \frac{2\pi}{T} = \frac{2e}{\hbar}\langle V \rangle$.
The current-voltage characteristic of a non-hysteretic ($C \to 0$) junction is found to be,
$\langle V \rangle = \begin{cases} 0 & I < I_c \\ IR_N\sqrt{1-(I_c/I)^2} & I > I_c \end{cases}$

If the junction has large capacitance C, it is analogous to a bowling ball running down the periodic potential under the influence of drag. The onset of voltage with increasing tilt will be the same as the previous ($C \to 0$) case. However, as the tilt is reduced in the running state, the inertia of the junction/bowling ball will prevent it from coming to rest at $I = I_c$ as the current is reduced. It will take a further reduction of the tilt to bring the phase point to rest. This results in a hysteretic response of the junction, and allows one to create a digital bit in which the JJ will have either zero voltage or a finite voltage for the same current, depending on the history of the device. This was the basis for Josephson logic used to make computers back in the 1980's. Nowadays the dissipation in the '1' state is too great to tolerate, so more subtle ways of representing '1's and '0's using Josephson junctions have been developed!

IV. DC AND AC CURRENT-BIASED JOSEPHSON JUNCTION

Now consider a Josephson junction with an ac current bias in addition to the dc bias. The total current through the JJ is split three ways in general (going through the ideal JJ, resistor, capacitor),
$I_{dc} + I_{ac}\sin(\omega_{ac}t) = I_c \sin\gamma + \frac{\Phi_0}{2\pi R_N}\frac{d\gamma}{dt} + C\frac{\Phi_0}{2\pi}\frac{d^2\gamma}{dt^2}$.
Multiply this current (I) equation by voltage ($\frac{\hbar}{2e}\frac{d\gamma}{dt}$) to get the instantaneous power equation as,
$\frac{d}{dt}\left\{\frac{1}{2}(\frac{\hbar}{2e})^2 C(\frac{d\gamma}{dt})^2 + \left[\frac{-\hbar I}{2e}\gamma - \frac{\hbar I_c}{2e}\cos\gamma\right]\right\} = \left[\frac{\hbar I_{ac}\sin(\omega_{ac}t)}{2e} - (\frac{\hbar}{2e})^2\frac{1}{R_N}\frac{d\gamma}{dt}\right]\frac{d\gamma}{dt}$.
The left side appears to be the time rate of change of kinetic energy plus potential energy, while the right hand side is the power dissipated in the resistor plus a fluctuating force provided by the AC current bias.
The ac bias current has two consequences:
1) The agitation of the washboard produced by the "ac wiggle" will induce the phase particle to jump over the barrier a little early as the washboard is tilted towards the critical current. Thus the critical current will be reduced slightly by this agitation.
2) Shapiro steps. Consider the junction in the running ($\frac{d\gamma}{dt} \neq 0$) finite-voltage state. As it "falls" along the tilted washboard, the fictitious phase particle will speed up and slow down periodically in time. The motion is perioidic, but not sinusoidal. On average the phase particle will cover 2π radians in a period T, resulting in an average angular frequency $\langle\frac{d\gamma}{dt}\rangle = 2\pi/T$.
A resonance condition can be satisfied when the ac drive frequency ω_{ac} coincides with the periodicity of the motion of the phase point $2\pi/T$. In this case the driving current source can "phase lock" with motion of the phase particle and there can be resonant absorption of energy by the JJ. Due to the intrinsic nonlinearity of the JJ, this may happen over a range of dc current values, resulting in a "Shapiro step" in the current-voltage characteristic. The first Shapiro step will occur at a voltage given by $\langle V_1 \rangle = \frac{\hbar\omega_{ac}}{2e}$.
As the washboard is tilted further (higher I_{dc}), the phase particle will move faster and it can cover multiple 2π periods of the washboard potential in the period of the ac drive. Hence there will be higher-order Shapiro steps, given by voltages $\langle V_n \rangle = n\frac{\hbar\omega_{ac}}{2e}$, with $n = 1, 2, 3....$ Plugging in the numbers, the voltage step size will be $V_n = n\frac{\nu}{483.6MHz/\mu V}$, where ν is the linear frequency of the ac current bias in MHz.
The class web site shows these Shapiro steps in the IV curve of a Nb point contact.